Zoological Studies 34, Supplement I (1995)

Zoological Studies

The Wigner Distribution Function, and the Special Affine Fourier Transform: Signal Processing and Optical Imaging

S. Abe[1] and J. T. Sheridan[2]
[1]College of Science and Technology, Nihon University, 7-24-1 Narashinodai, Chiba 274, Japan
[2]European Commision, Joint Research Centre (JRC), Institute for Systems Engineering and Informatics, (ISEI), Ispra (VA), I-21020, Italy

Signal analysis is usually performed in the space (time) or spatial-frequency (frequcncy) domains through the classical Fourier transformation (FT) (Goodman 1994, Stremler 1982). Since these two domains are orthogonal to each other, information on the spatial position is completely lost when working in the spatial-frequency domain, or vice versa. However, there are situations in which it is still desirable to get information simultaneously on the position and spatial frequency of a signal. Such information is limited by the uncertainty principle.

The Wigner distribution function (WDF) (Wigner 1932) offers the representation of a signal function in terms of position and spatial-frequency variables, called the phase-space variables. (x , k) simultaneously.

Recently the concept of the fractional Fourier transformation (FRT), (Namias 1980, McBride and Kerr 1987) has been introduced to the area of optical signal processing, (Lohmann 1993, Ozaktas et al. 1994, Dorsch et al. 1994). It has been pointed out that a GRIN medium can be used to directly produce the FRT of an input field by varying its length. The FRT corresponds to the rotation of the WDF in the phase space, (Lohmann,1993).

Geometrically, in phase space, the classical FT is given by a (Pi)/2 rotation. Since the FRT allows an arbitrary rotation angle, it can avoid the above mentioned difficulty concerning the orthogonality between the two domains. Recent investigations on signal analysis based on the FRT shows how effectively the noise filtering can be realized, (Ozaktas et al. 1994, Dorsch et al. 1994). Optical generations of the FRT, using free space propagation and a lens (es) has also been proposed, (Lohmann 1993).

Here we first wish to discuss a more general class of the Fourier-type transformations. We consider the special affine transformation in the phase space, (i. e., the most general power-conserving linear transformation) and present the corresponding integral transformation of the signal function. It is referred to as the special affine Fourier transformation (SAFT), (Abe and Sheridan 1994a, Abe and Aheridan 1994b, Abe and Sheridan 1994c, Abe and Sheridan 1995). In this formulation, the FRT is the simple transformational sub-case of the rotation.

Then we apply the SAFT to standard operations in geometric optics. We show the SAFT interrelates geometric optics operations and the corresponding wave-function transformations. The operations of lens, free-space propagation, magnification, and the FRT are seen to be he Abelian subgroups of the SAFT, (Abe, Sheridan, 1994b). Next we apply the SAFT to an optical system which has small imperfections, to establish the concept of the almost-Fourier and almost-Fresnel transformations, (Abe and Sheridan 1994c).

We briefly discuss the filtering problem based on the SAFT. Such a mathematical formalism can offer an unified approach to local signal analysis. In particular, we show that the wavelet and windowed Fourier transformations are included as special cases.

Finally the FRT has also been shown to provide a new means of interpreting and analysing afocal imaging systems, (Bernardo and Soares 1994). Once again the SAFT allows more general ideas to emerge in this area.

REFERENCES

Abe S, JT Sheridan. 1994a. Generalisation of the fractional Fourier transformation to an arbitrary lineatr lossless transformation: An operator approach. J. Phys. A, Math. Gen. **27:** 4179-4187.

Abe S, JT Sheridan. 1994b. Optical operations on wave functions as the Abelian subgroups of the special affine Fourier transformation. Opt. Lett., **9:** 1801-1803.

Abe S, JT Sheridan. 1994c. Comment on "The fractional Fourier transformation in optical propagation problems". J. Mod. Opt., Submitted and under review.

Abe S, JT Sheridan. 1995. The almost-Fourier and almost-Fresnel transformations, Opt. Comm. **133:** 358-388.

Bernardo L M, ODD Soares. 1994. Fractional Fourier transforms and imaging, J, Opt. Soc. Am. **11:** 2622-2626.

Dorsch RG, AW Lohmann, Y Bitran, D Mendlovic, HM Ozaktas, 1994. App. Opt. **33:** 7599.

Goodman JW. 1994. Introduction to Fourier optics, 2nd Edition, McGraw-Hill Co., Inc.

Lohmann A. 1993. Image lotation, Wigner rotation, and the fractional Fourier transform. J. Opt. Soc. Am. A. **10:** 2181.

McBride AC, FH Kerr. 1987. On Namias's fractional Fourier transform. Inst. Math Appl. J. Appl. **25:** 159-179.

Namias V. 1980. The fractional order Fourier transform and its applications to quantum mechanicw. J. Inst. Math. Appl. **25:** 241-165.

Ozaktas H M, B Barshan, D Mendlovic, D Onural. 1994. Convolution, filtering and multiplexing in fractional Fourier domains and their relation to chirp and wavelet transforms. J. Opt. Soc. Am A. **11:** 522-559.

Stremler FG. 1982. Introduction to communication systems, 2nd Edition, Addison-Wesley, Reading, Massachusetts.

Wigner E. 1932. On the quantum correction for the thermodynamic equilibrium. Phys. Rev. **40:** 749-759.